State-Feedback Switching Control for Discrete-Time Takagi-Sugeno Fuzzy Systems Based on Partitioning The Range of Fuzzy Weights

Won Il Lee, Jeong Wan Ko, and PooGyeon Park

***Abstract*—In this paper, we propose an efficient relaxation method of the parameterized linear matrix inequalities (PLMIs) in the framework of the state-feedback stabilization problem for discrete-time Takagi-Sugeno (T–S) fuzzy systems. The matrix elimination method plays a key role in deriving the criterion, which reduces the order of the fuzzy weights by eliminating the quadratic fuzzy weights in the original PLMIs and then transformed to a more tractable one. A partition on the range of the fuzzy weights is introduced, through which a linearly weight-dependent condition can be developed by fixing some decision variables piecewisely. By utilizing the extreme points of each partition, the negativity of the condition can be guaranteed and the corresponding controller is represented in the form of a switching control law based on the partition. Some example shows that finer subdivision in the partition leads to a better performance behavior.**

***Index Terms*—Fuzzy systems, state-feedback, switching control, stabilization.**

I. Introduction

In recent decades, T–S fuzzy control systems have attracted considerable attention from academic research and industrial applications in virtue of its versatility to model many complex nonlinear systems as a tractable one [1], and a number of stability analysis and controller design results have been reported in the literature [2]-[6].

The development of controllers in this field has been with the efforts to exploit the information of the fuzzy weights, utilizing as much information as possible to the control gains. Initially, the conventional constant control gain was modified to be linear in the fuzzy weights, say the parallel distributed compensation (PDC) law [5]. By employing a common quadratic Lyapunov function $V(t) = x^T(t)Px(t),\ P > 0,$ a quadratically weight-dependent PLMIs are obtained, whose negativity was confirmed by the vertex-wise method. As a way of reducing conservatism, [4] generalized the linearly weight-dependent gains to be nonlinear and adopted a nonlinear Lyapunov function, say the non-PDC law. Here a main concern is how to relax the complicated formulation to a manageable one. Though, [3] introduced an interesting relaxation method for this, there still seems to be room for further improvement. Anyway, approaches based on the non-PDC fuzzy controllers offer higher design flexibility and show much better performance behavior than the PDC ones.

In this paper, instead of focusing on designing the structure of the Lyapunov function or the control gains, we shall pay attention to relaxation methodologies of the resulting PLMIs that are quadratic in the fuzzy weighting functions. The quadratic PLMIs are transformed to a more tractable form by introducing some decision variables via the matrix elimination lemma [7]. Then, the range of the fuzzy weights is partitioned properly and the decision variables are fixed piecewisely in accordance with the partition to derive a condition that is linearly dependent on the fuzzy weights, so that the negativity of the final condition can be guaranteed by utilizing the extreme points of each partition from the polyhedral optimization perspective [8], and a switching controller based on the partition can be developed.

The proposed relaxation method has the edge over the existing ones in many perspectives. It is relatively simple and can be applied to almost all the previous results in the literature. By elaborating the partition on the range of the fuzzy weights, its performance can be improved further. Actually, we validate in Section III that finer subdivision in the partition leads to a better performance behavior. Another merit of the extreme points approach is that the bounds of the fuzzy weights can be easily handled just by imposing additional conditions to the conventional linear programming tools.

The paper is organized as follows. Section II proposes a state-feedback stabilization criterion for discrete-time T–S fuzzy systems via a switching control scheme. In Section III, a simple numerical example is given to verify the effectiveness of the proposed method.

Notation: $\left[a_i + b\right] = \begin{bmatrix} a_1 + b \\ \vdots \\ a_r + b \end{bmatrix},\ \left[a_{i,j} + b\right] = \begin{bmatrix} a_{1,1} + b & \cdots & a_{1,r} + b \\ \vdots & \ddots & \vdots \\ a_{r,1} + b & \cdots & a_{r,r} + b \end{bmatrix}$

II. Main Result

A. State-Feedback Stabilization

Let us consider the following discrete-time T–S fuzzy system:

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$$x(k+1) = A(k)x(k) + B(k)u(k),$$

$$[A(k) \;\; B(k)] = \sum_{i=1}^{r} \theta_i(k)[A_i \;\; B_i], \tag{1}$$

where, $x(k) \in \mathbf{R}^{n_x}$ is the state and $u(k) \in \mathbf{R}^{n_u}$ is the control input. Here, $\theta_i(k)$ denote the normalized fuzzy weighting functions in the range

$$\Psi = \left\{ (\theta_1(k), \cdots, \theta_r(k)) \mid \sum_{i=1}^{r} \theta_i(k) = 1, 0 \le \alpha_i(k) \le \theta_i(k) \le \beta_i(k) \right\}, \tag{2}$$

where r is the number of fuzzy rules.

Under the state-feedback control law:

$$u(k) = K(k-1,k)x(k), \tag{3}$$

And the non-quadratic Lyapunov function [4]:

$$V(k) = x^T(k)G^{-T}(k-1)P(k-1)G^{-1}(k-1)x(k), \tag{4}$$

where $P(k-1) > 0$, the following Lyapunov derivative condition for asymptotic stability of the closed-loop system is obtained:

$$0 > \{A(k)G(k-1) + B(k)\bar{K}(k-1,k)\}^T G^{-T}(k)P(k)G^{-1}(k)$$
$$\times \{A(k)G(k-1) + B(k)\bar{K}(k-1,k)\} - P(k-1), \tag{5}$$

where $\bar{K}(k-1,k) = K(k-1,k)G(k-1)$, which can be relaxed with the help of [3] as

$$\begin{pmatrix} P(k-1) & * \\ A(k)G(k-1) + B(k)\bar{K}(k-1,k) & G(k) + G^T(k) - P(k) \end{pmatrix} > 0 \tag{6}$$

Or equivalently

$$0 < e_1 P(k-1)e_1^T + e_2(G(k) + G^T(k) - P(k))e_2^T$$
$$+ e_1(A(k)G(k-1) + B(k)\bar{K}(k-1,k))^T e_2^T$$
$$+ e_2(A(k)G(k-1) + B(k)\bar{K}(k-1,k))e_1^T \tag{7}$$

where $e_1 = [I \quad 0]^T \in \mathbf{R}^{2n_x \times n_x}$ and $e_1 = [0 \quad I]^T \in \mathbf{R}^{2n_x \times n_x}$.

B. *Switching Control Based on Partitioning the Range of Fuzzy Weights*

Let us define

$$\bar{\theta}_{i,1\le i\le 2r} = \begin{cases} \theta_i(k-1), & 1 \le i \le r \\ \theta_{i-r}(k), & r+1 \le i \le 2r \end{cases},$$

$$\bar{E} = \frac{1}{2}[I \quad \cdots \quad I]^T \in \mathbf{R}^{2r\cdot 2n_x \times 2n_x},$$

$$\bar{E}_1 = [I \quad 0]^T \in \mathbf{R}^{2r\cdot 2n_x \times r\cdot 2n_x}, \bar{E}_2 = [0 \quad I]^T \in \mathbf{R}^{2r\cdot 2n_x \times r\cdot 2n_x},$$

$$\bar{\Theta}(k) = [\theta_1(k-1)I \quad \cdots \quad \theta_r(k-1)I \quad \theta_1(k)I \quad \cdots \quad \theta_r(k)I]^T$$
$$\in \mathbf{R}^{2r\cdot 2n_x \times 2n_x}. \tag{8}$$

Note that if (7) can be rewritten as

$$0 > \bar{\Theta}^T(k)\Omega(\bar{\Theta}(k))\bar{\Theta}(k) \tag{9}$$

Then, by the elimination lemma [7] and the relation $(I - \bar{E}\bar{\Theta}^T(k))^{\perp} = \bar{\Theta}(k)$ from the fact that

$$\text{rank}\left[I - \bar{E}\bar{\Theta}^T(k) \quad \bar{\Theta}(k)\right] \ge \text{rank}\left[I - \bar{E}\bar{\Theta}^T(k) \quad \bar{\Theta}(k)\right]\begin{bmatrix} I \\ \bar{E}^T \end{bmatrix},$$

$(I - \bar{E}\bar{\Theta}^T(k) + \bar{\Theta}(k)\bar{E}^T)^T + (I - \bar{E}\bar{\Theta}^T(k) + \bar{\Theta}(k)\bar{E}^T) > 0$, (9) holds if and only if there exists $\Pi(\bar{\Theta}(k))$ such that

$$0 > \Omega(\bar{\Theta}(k)) + \Pi(\bar{\Theta}(k))(I - \bar{\Theta}(k)\bar{E}^T)$$
$$+ (I - \bar{\Theta}(k)\bar{E}^T)^T \Pi^T(\bar{\Theta}(k)) \tag{10}$$

For each $(\theta_1(k), \cdots, \theta_r(k)) \in \Psi$ with an admissible transition from $(\theta_1(k-1), \cdots, \theta_r(k-1)) \in \Psi$. Since it is difficult to find directly $\Pi(\bar{\Theta}(k))$, let us partition Ψ into $\bigcup_{h=1}^{S} \Psi_h$ and design as $\Pi(\bar{\Theta}(k)) = \Pi_{h(k-1),h(k)}$ for $(\theta_1(k), \cdots, \theta_r(k)) \in \Psi_{h(k)}$ transited from $(\theta_1(k-1), \cdots, \theta_r(k-1)) \in \Psi_{h(k-1)}$. Assuming that Ψ_h is a polyhedral subset of $\mathbf{R}^r$ and

$$[P(k) \; G(k) \; \bar{K}(k-1,k)] = [\sum\nolimits_{i,j}^{r} \theta_i(k)\theta_j(k)P_{i,j}^{h(k)}$$
$$\sum\nolimits_{i,j}^{r} \theta_i(k)\theta_j(k)G_{i,j}^{h(k)} \; \sum\nolimits_{i,j}^{2r} \bar{\theta}_i(k)\bar{\theta}_j(k)\bar{K}_{i,j}^{h(k-1),h(k)}] \tag{11}$$

Lead to the following theorem.

Theorem 1 : For a given polyhedral partition $\Psi = \bigcup_{h=1}^{S} \Psi_h$, the switching control law

$u(k) = \{\bar{K} - 1, k)G^{-1}(k-1)x(k) \mid$
$(\theta_1(k-1), \cdots, \theta_r(k-1)) \in \Psi_{h(k-1)}, (\theta_1(k), \cdots, \theta_r(k)) \in \Psi_{h(k)}\}$

asymptotically stabilizes the T–S fuzzy system (1) if there exist matrices $P_{i,j}^a, P_{i,j}^b, G_{i,j}^a, G_{i,j}^b, \bar{K}_{f,g}^{a,b}$, and $\Pi_{a,b}$ such that the following inequality

$$0 < \bar{\Omega}(\bar{\Theta}^{a,b}) + \bar{\Omega}^T(\bar{\Theta}^{a,b}) + \bar{E}_1[e_1 P_{i,j}^a e_1^T]\bar{E}_1^T$$
$$-\bar{E}_2^T[e_2 P_{i,j}^b e_2^T]\bar{E}_2^T, \quad [P_{i,j}^a]^T = [P_{i,j}^a] \tag{12}$$

Holds for all $1 \le i, j \le r, 1 \le f, g \le 2r$, and all the extreme points $(\theta_1^a, \cdots, \theta_r^a) \in \Psi_a, (\theta_1^b, \cdots, \theta_r^b) \in \Psi_b$ in $1 \le a, b \le S$ with all admissible transitions from Ψ_a to where Ψ_b, and

$$\bar{\Omega}(\bar{\Theta}^{a,b}) = \bar{E}_1[e_2 \sum\nolimits_n^r \theta_n^b A_n G_{i,j}^a e_1^T]\bar{E}_1^T + [e_2 \sum\nolimits_n^r \theta_n^b B_n \bar{K}_{f,g}^{a,b} e_1^T]$$
$$+ \bar{E}_2[e_2 G_{i,j}^b e_2^T]\bar{E}_2^T + \Pi_{a,b}(I - \bar{\Theta}^{a,b}\bar{E}^T).$$

Proof : Assume $(\theta_1(k-1), \cdots, \theta_r(k-1)) \in \Psi_{h(k-1)}$ and $(\theta_1(k), \cdots, \theta_r(k)) \in \Psi_{h(k)}$. Since (12) is convex in $\bar{\theta}_i$, (12) over the extreme points of (Ψ_a, Ψ_b) implies [8]

$$0 < \bar{\Omega}(\bar{\Theta}(k)) + \bar{\Omega}^T(\bar{\Theta}(k)) + \bar{E}_1[e_1 P_{i,j}^{h(k-1)} e_1^T]\bar{E}_1^T$$

$$-\overline{E}_2^T[e_2 P_{i,j}^{h(k)} e_2^T]\overline{E}_2^T.$$

This leads to the condition (10) and also (9). Note that (7) guarantees positivity of $P(k)$ and invertibility of $P(k)$.

III. Example

The extreme points of each polyhedral partition were obtained through the open source 'extrpts.m' in MATLAB Central.

Consider the discrete-time T–S fuzzy model in [4]:

$$\begin{aligned} A_1 &= \begin{bmatrix} 1 & -\beta \\ -1 & -0.5 \end{bmatrix}, B_1 = \begin{bmatrix} 5+\beta \\ 2\beta \end{bmatrix}, \\ A_2 &= \begin{bmatrix} 1 & \beta \\ -1 & -0.5 \end{bmatrix}, B_2 = \begin{bmatrix} 5-\beta \\ -2\beta \end{bmatrix}. \end{aligned} \tag{13}$$

The following polyhedral partitions:

$$\Psi^1 = \{\theta_i \mid 0 \le \theta_i \le 1\},$$
$$\Psi_1^2 = \{\theta_i \mid \theta_1 \le \theta_2\},\ \Psi_2^2 = \{\theta_i \mid \theta_2 \le \theta_1\}.$$

As well as the boundary conditions below are considered:

$$\Psi^3 = \{\theta_i \mid 0.1 \le \theta_i \le 0.9\}.$$

The maximum value of β that guarantees asymptotic stabilization of the system is checked and summarized in Table I. It shows that finer subdivision in the partition leads to a better performance behavior.

TABLE I: Maximum β of The Stabilizable Region

Ψ =	Ψ^1	Ψ^2	Ψ^3
[9, Theorem 6 (N=2)]	1.7523	·	·
[10, Theorem 6]	1.7659	·	·
[4, Theorem 5]	1.7666	·	·
[11, Theorem 4]	1.7821	·	·
[12, Theorem 1 (N=2)]	1.7824	·	·
Theorem 1	1.8043	1.8281	2.2553

IV. Conclusion

This paper concerned the state-feedback stabilization problem for discrete-time T–S fuzzy systems by introducing an efficient relaxation method of the quadratically parameterized LMIs. By partitioning the range of the fuzzy weights, a switching controller based on the partition was proposed. The simulation example showed that finer subdivision in the partition leads to a better performance behavior and also showed that our method can handle the boundary constraints on the fuzzy weights efficiently.

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